2.19

$$\lim_{u\to 0} \frac{e^{u^2}-1}{\cos(3u)-1} = \lim_{u\to 0} \frac{e^{u^2}-1}{u^2} \cdot \frac{u^2}{\cos(3u)-1}$$

1

$$= \lim_{u\to 0} \frac{u^2(\cos(3u)+1)}{(\cos(3u)-1)(\cos(3u)+1)}$$

$(A-B)(A+B) = A^2-B$

$$= \lim_{u\to 0} \frac{u^2(\cos(3u)+1)}{\cos^2(3u)-1}$$

$-\sin^2(3u)$

$\cos^2(\theta)+\sin^2(\theta)=1$

$\cos^2\theta - 1 = -\sin^2\theta$

$$= \lim_{u\to 0} \frac{u^2(\cos(3u)+1)}{-\sin^2(3u)}$$

$$= \lim_{u\to 0} \frac{-u^2(\cos(3u)+1)}{1} \cdot \frac{\cancel{(3u)^2}}{\sin^2(3u)\cancel{(3u)^2}}$$

$\left(\frac{3u}{\sin(3u)}\right)^2 \to 1$

$$= \lim_{u\to 0} \frac{-u^2(\cos(3u)+1)}{(3u)^2}$$

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

$$= \lim_{u\to 0} \frac{-\cancel{u^2}(\cos(3u)+1)}{9\cancel{u^2}} \qquad \cos(3u)+1 \to 2$$

$$= \boxed{-\frac{2}{9}}.$$

$$\lim_{\theta\to 0} \frac{\theta}{\sin\theta} = \lim_{\theta\to 0} \frac{1}{\frac{\sin\theta}{\theta}} \qquad \frac{\sin\theta}{\theta} \to 1$$

$$= \frac{1}{1} = 1$$

$$\lim_{t\to 0^-} e^{\frac{1-t}{t}} = ?$$

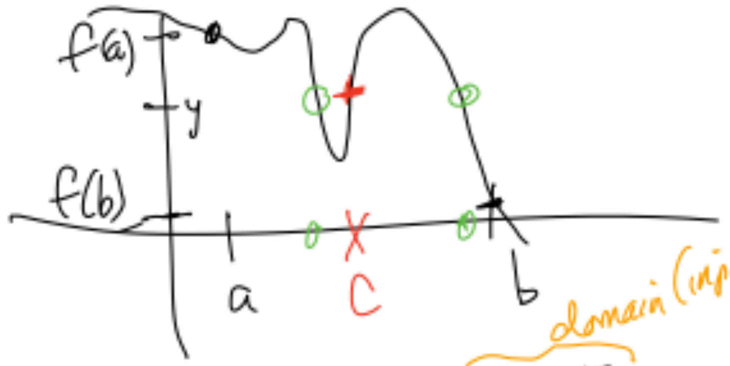
$$\left(\lim_{t\to 0^-} \frac{1-t}{t} = -\infty \right) \qquad 1-t \to 1^+, \quad t \to 0^-$$

$$\lim_{t\to 0^-} e^{\frac{1-t}{t}} = \boxed{0} \qquad \frac{1-t}{t} \to -\infty$$

$$e^{-\text{large \#}} = \frac{1}{e^{\text{large \#}}}$$

Last minute thoughts about limits & continuity

① Intermediate Value Theorem



If $f : \overbrace{[a,b]}^{\text{domain (inputs)}} \to \overbrace{\mathbb{R}}^{\text{codomain (where outputs live)}}$ is a continuous function, and if y is between $f(a)$ and $f(b)$, then there exists $\underbrace{(\exists)}_{\text{means "there exists"}}$ $c \underbrace{\in}_{\text{means "in"}} (a,b)$ such that $f(c) = y$.

Formal definition of limit.

We say $\lim_{x \to a} f(x) = L$ if

$\overset{\text{"for all"}}{\forall} \varepsilon > 0$, $\exists \delta > 0$ such that

if $\underbrace{0 < |x-a| < \delta}_{x \text{ is getting close to } a \text{ but } \neq a.}$ and $x \in \text{domain}(f)$,

then $\underbrace{|f(x) - L| < \varepsilon}_{f(x) \text{ is getting really close to } L}$.

Tangent lines & secant lines

Example – Find the equation of the tangent line to $y=x^2-3x+1$ at the point where $x=\frac{1}{2}$

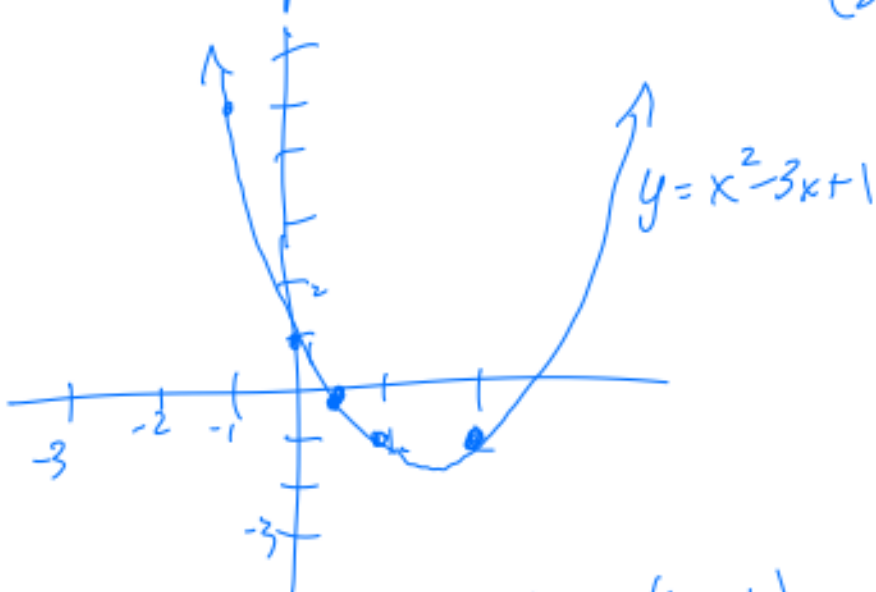
$\left(\frac{1}{2}\right)^2-3\left(\frac{1}{2}\right)+1$

$\frac{1}{4}-\frac{3}{2}+1$

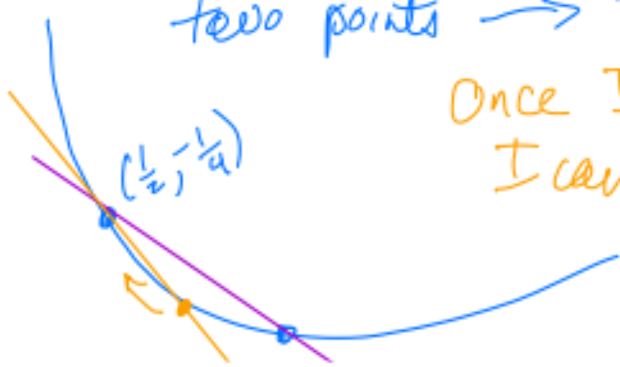
$=\frac{1}{4}-\frac{6}{4}+\frac{4}{4}$

$=\frac{-1}{4}$

$1^2-3\cdot1+1=-1$



Strategy: pick one pt as $\left(\frac{1}{2},\frac{-1}{4}\right)$ on the curve.
pick other points on the curve close to that
Find the slope of the line between the two points → take a limit.

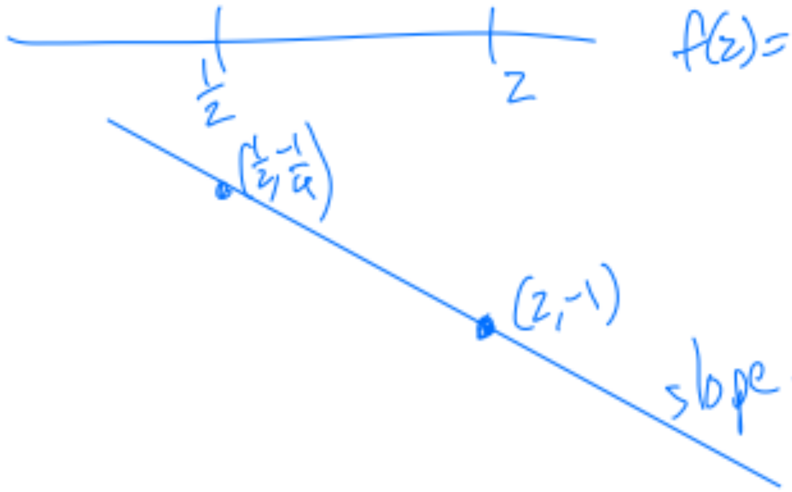


Once I have the slope, I can find the equation of the line, using

$$y - y_0 = m(x - x_0)$$

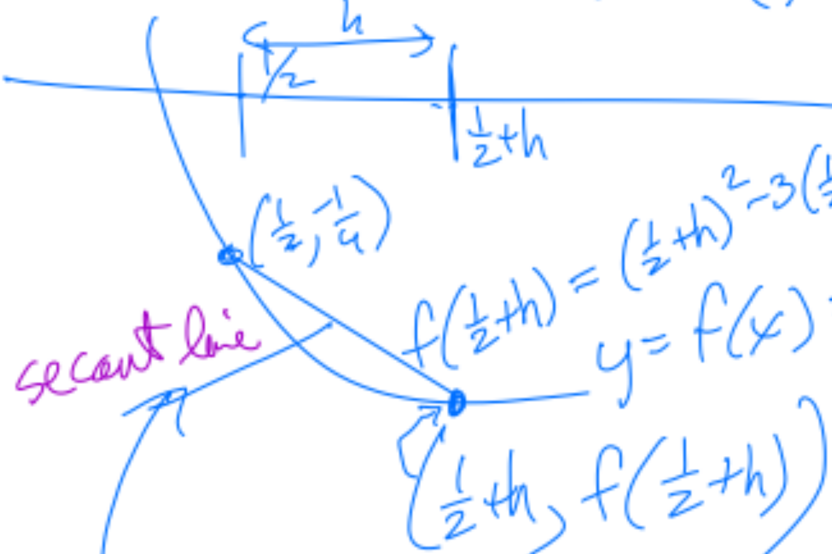
$$f(x) = x^2 - 3x + 1$$

$$f(2) = 2^2 - 3\cdot 2 + 1 = -1$$



$$\text{slope} = \frac{-1 - \left(-\frac{1}{4}\right)}{2 - \frac{1}{2}} = \frac{-3/4}{3/2} = -\frac{3}{4}\cdot\frac{2}{3} = \boxed{-\frac{1}{2}}$$

Let's find a ton of slopes starting at $\left(\frac{1}{2}, -\frac{1}{4}\right)$.



$$f\left(\tfrac{1}{2}+h\right) = \left(\tfrac{1}{2}+h\right)^2 - 3\left(\tfrac{1}{2}+h\right) + 1$$

$$y = f(x) = x^2 - 3x + 1$$

$$\text{slope} = \frac{f\left(\frac{1}{2}+h\right) - \left(-\frac{1}{4}\right)}{\left(\frac{1}{2}+h\right) - \frac{1}{2}} = \frac{f\left(\frac{1}{2}+h\right) + \frac{1}{4}}{h}$$

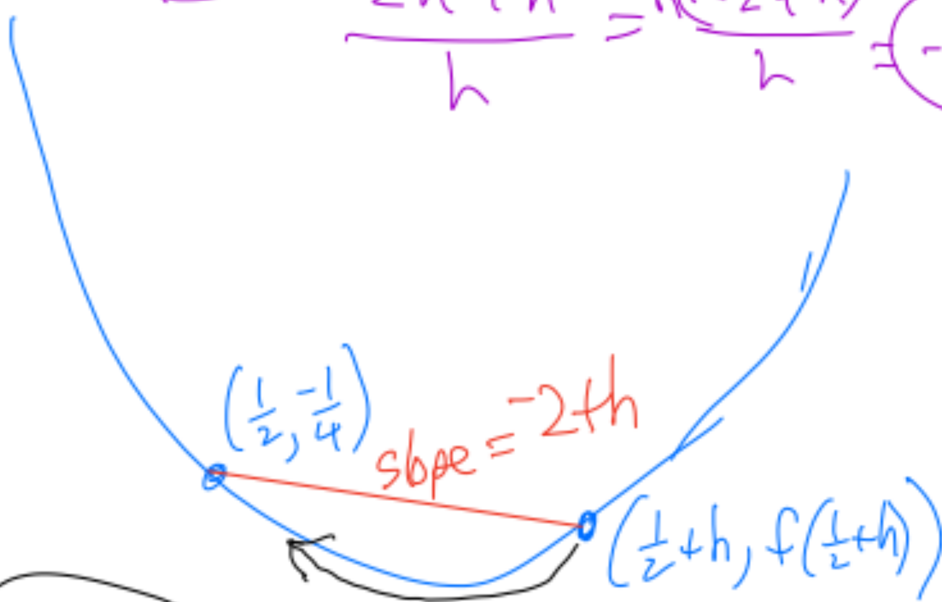
$$\text{slope} = \frac{f\left(\frac{1}{2}+h\right)+\frac{1}{4}}{h} = \frac{\left(\frac{1}{2}+h\right)^2 - 3\left(\frac{1}{2}+h\right)+1+\frac{1}{4}}{h}$$

$$\text{slope} = \frac{\frac{1}{4}+h+h^2-\frac{3}{2}-3h+\frac{5}{4}}{h}$$

add to 0

$$\frac{1}{4}-\frac{6}{4}+\frac{5}{4}=0$$

$$= \frac{-2h+h^2}{h} = \frac{h(-2+h)}{h} = -2+h$$



$$\text{tangent slope at } x=\tfrac{1}{2} = \lim_{h\to 0} \text{slope}(h)$$

$$= \lim_{h\to 0}(-2+h) = \boxed{-2}$$

tangent slope

equation of the line is

$$y-\left(\frac{-1}{4}\right)=-2\left(x-\frac{1}{2}\right)$$

$$y+\frac{1}{4}=-2x+1$$

$$\boxed{y=-2x+\frac{3}{4}}$$

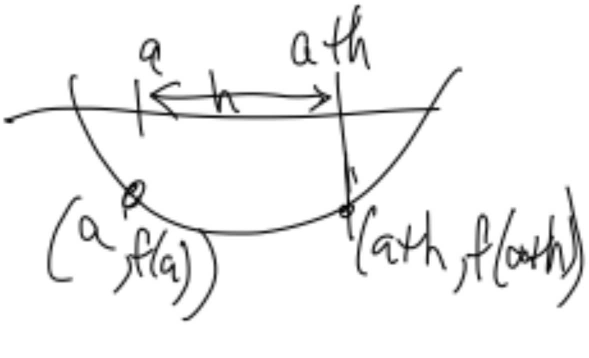
What we have done, in general:

Given $y=f(x)$ curve

$x=a$ gives point $(a, f(a))$ on the curve.

$$\text{tangent slope at } x=a = \lim_{h\to 0}\frac{f(a+h)-f(a)}{a+h-a}$$

$$= \boxed{\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}}$$



$\boxed{y-y_0=m(x-x_0) \text{ eqn of line}}$

$$\frac{\Delta y}{\Delta x}$$

Examples

① Find the tangent slope & equation of the tangent line to $y = x^4 - 7x + 8$ at $(1,2)$.

$f(x)$

$$\text{slope} = \frac{f(a+h) - f(a)}{h} = \frac{(a+h)^4 - 7(a+h) + 8 - 2}{h}$$

$1+h$

$a = 1$

$$(1+h)^4 = 1^4 + 4 \cdot 1^3 h + 6 \cdot 1^2 h^2 + 4 \cdot 1 \cdot h^3 + h^4$$

$$= 1 + 4h + 6h^2 + 4h^3 + h^4$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

0

$$\text{slope} = \frac{1 + 4h + 6h^2 + 4h^3 + h^4 - 7 - 7h + 8 - 2}{h}$$

$$\text{slope} = \frac{-3h + 6h^2 + 4h^3 + h^4}{h}$$

$$\text{slope} = \frac{\cancel{h}(-3 + 6h + 4h^2 + h^3)}{\cancel{h}}$$

$$\text{slope} = -3 + 6h + 4h^2 + h^3$$

$$\text{tangent slope} = \lim_{h \to 0} \left(-3 + 6h + 4h^2 + h^3\right)$$

$$= \boxed{-3}.$$

point $(1,2)$

$$y - 2 = -3(x-1)$$

$$y - 2 = -3x + 3$$

$$\boxed{y = -3x + 5}$$

Example Find the tangent line to $\underbrace{y = \cos\left(2x - \frac{\pi}{2}\right)}_{f(x)}$ at $x = \frac{\pi}{4}$.

$$\begin{pmatrix}\text{slope} \\ \text{secant} \\ \text{line}\end{pmatrix} = \frac{f\left(\frac{\pi}{4} + h\right) - f\left(\frac{\pi}{4}\right)}{h}$$

$$f\left(\frac{\pi}{4}\right) = \cos\left(2\left(\frac{\pi}{4}\right) - \frac{\pi}{2}\right)$$
$$= \cos(0) = 1$$

$$\text{slope} = \frac{\cos\left(2\left(\frac{\pi}{4}+h\right) - \frac{\pi}{2}\right) - 1}{h}$$

$$= \frac{\cos\left(\frac{\pi}{2} + 2h - \frac{\pi}{2}\right) - 1}{h}$$

$$= \frac{\cos(2h) - 1}{h}$$

tangent Slope

$$= \lim_{h \to 0} \frac{(\cos(2h) - 1)}{h} \frac{(\cos(2h)+1)}{(\cos(2h)+1)}$$

$\frac{0}{0}$ form

$$= \lim_{h \to 0} \frac{\cos^2(2h) - 1}{h(\cos(2h)+1)}$$

$\cos^2(2h) - 1 = -\sin^2(2h)$

$$= \lim_{h \to 0} \frac{-\sin^2(2h)}{h(\cos(2h)+1)}$$

$$= \lim_{h \to 0} \frac{-1}{h(\cos(2h)+1)} \cdot \frac{\sin^2(2h)}{(2h)^2} \cdot (2h)^2$$

$$\left(\frac{\sin\theta}{\theta}\right)^2 \to 1$$

$$= \lim_{h \to 0} \frac{-4h^{\cancel{2}}}{\cancel{h}\underbrace{(\cos(2h)+1)}_{2}} = \frac{0}{2} = \boxed{0}$$

tan slope $= 0$

$$y - y_0 = 0(\cancel{x} - x_0)$$

$$y - 1 = 0$$

$\boxed{y = 1}$ tangent line